

The laws of Newton and Coulomb as information transmission by virtual particles

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Abstract

In elementary particle physics the philosophy of virtual particles is widely used. We use this philosophy to obtain the famous inverse square law of classical physics. We define a formal model without fields or forces, but with virtual particle - information transmitter. This formal model admits very simple (school level) interpretation with two classical particles and one virtual. Then we prove (in a mathematically rigorous way) that the trajectories in our model converge to standard Newtonian trajectories of classical physics.

1 Introduction

The main result is formulated in the title of the paper. Although it has connections with various fields of mathematics and mathematical physics, I could not find a unique best framework for it. Therefore, this needs serious comments, that we do in this introduction.

Dynamical systems theory We consider here the formal system of recurrent equations (2-4), which defines strongly non-linear dynamical system in three dimensional space. Similar iterations of rational functions were studied by many authors, and are normally sufficiently difficult [8]. However, our problems are different, and we do not use the results of this big science. The goal of this paper is to get closed results for some scaling of parameters.

Classical particle physics Gravitational and electric forces, that describe so different physical phenomena, surprisingly have the same form - inverse square law, differing only by constant factors. These laws are related to harmonic functions and the Poisson equation. Already long ago there exist other - geometric - approaches to the Kepler laws, see for example [2, 3]. But more important, it appeared possible to deduce the Newton and Coulomb laws from more general (and more complicated) physical theories - general relativity theory of Einstein and Maxwell's electrodynamics, correspondingly, where the fields play basic role. Here we show connection of these laws with quite different models.

Numerical methods Computational methods in physics are quite developed, and it might seem that the formal system (2-4) is just an example of such computational schemes (for the Newton equations). However, this is not quite true, for two reasons:

1. one needs the number of steps of order c to reach times of order 1, and the main parameter c can be too large;

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2. but the main difference is that the time steps $\Delta_n t$ form one recurrent system with coordinates and velocities.

About information transmission in classical and quantum physics In non relativistic classical physics one has the action at a distance principle. For example, any particle with non-zero mass or non-zero charge influences on all space around itself. Otherwise speaking, information about this particle there is at the same time moment at any space point. In relativistic quantum physics Lienard-Wiechert potentials produce the same effect but with time delay.

In non-relativistic quantum mechanics (Schrodinger equation) the particles are quantum, but the fields are classical. That is why only fields act on the particles (action at a distance) but not vice-versa. About possibility of interpretation of quantum mechanics as information transmission see in [6].

From another side, philosophy of modern quantum field theory removes the difference between fields and particles, At the same time the Hamiltonians have terms of only two types - either defining the free movement of particles or (ultralocal interaction at one space point) instantaneous “chemical” reaction - transformation of one group of particles into another.

However, in elementary particle physics another principle is widely used - any two particle interaction uses a virtual particle - information transporter (called photon, graviton, ...). However, such conception has not yet gained the status of rigorous mathematics. In particular, the contradiction with the energy conservation law is explained with “energy-time” uncertainty principle, also beyond any athematical formulations. And the natural question arises whether it is possible to extend the idea of virtual particles - information transporters - on classical physics.

We demonstrate such possibility, at least in one-dimensional case. Namely, the particle trajectories in classical (non-relativistic) physics, which, as it is common to believe, are governed by the fundamental laws of Newton and Coulomb, can be obtained without introducing fields and forces, but only using particles - real and virtual. The interpretation of this model is extremely simple (on the school level), but the formulas look sufficiently difficult. That is why the proof demands direct analysis of these formulas as well. Moreover, surprisingly, the basis of this model is a simple relation between energy (or force) and time, see e.g. formula (12).

The interpretations of this system (2-4) cannot be now verified with any experiment. But the corollaries are the well-known physical laws. Concerning such situation the well-known physicist G. 't Hooft wrote in [1]: “Physicists investigating space, time and matter at the Planck scale will probably have to work with much less guidance from experimental input than has never happened before in the history of Physics. This may imply that we should insist on much higher demands of logical and mathematical rigor than before ...”. That is why the mathematical rigor obviously was the main goal of this paper.

On the information transmission in information computer networks Cellular automata, computer, communication, neural etc. networks are normally presented as a graph with the set V of vertices. In any vertex v of this graph there is an automaton, characterized at any time moment t by its state - the vector $x_v(t)$. Time is normally discrete and simultaneous transformation of all vectors $x_v(t)$ is defined by some functions

$$x_v(t+1) = f_v(x_w(t), w \in V)$$

providing the automata the information (prescription) what to do at the next moment. An important example are neural networks where the state of any neuron (vertex of the graph, or a particle on our language) can be either 1 or 0.

This principle of discreteness and simultaneity is often too rough approximation. Even in the habitual life (post, telegraph, telephone etc.) the information is transferred directionally through definite canal, and the time of information arrival is very important. Not only because of the queues but also because of the necessary synchronization, see for example [7].

On the other hand, many concrete physical phenomena (for example in kinetics) are modeled as a system of cellular automata - deterministic and stochastic, see for example [4]. Here the physical laws are, as though, approximated by automata systems, either giving a new class of numerical methods, or a class of beautiful mathematical analogies. All these aspects are reviewed in the fundamental monograph [5], see also [4, 6].

At the same time, in theoretical physics one can see frequent attempts to construct fundamental physical theories with discrete space-time, or even without any space-time. And thus a network graph seems to be the unique alternative to space, and time does not flow abstractly and independently of life, but depends on the state of the network. Although our model admits various interpretations with common one-dimensional space and standard time, but in fact the time for the main particle emerges as a corollary of the state of all system.

It is important to note that, during the deduction of inverse square, law the dimension 3 has never been used. At the same time to deduce it from Maxwell equations (or Poisson equation) the dimension 3 is absolutely necessary.

As a hypothesis we also note that our model can probably be generalized for any dimension and any number of particles.

2 Model and the main result

Formal definition of the model The model, that we denote by $M_0(c, \gamma)$, has two parameters $c > 0, \gamma > 0$, and three sequences of numbers $t_n, w_n, y_n, n = 0, 1, 2, \dots$, defined by the initial conditions

$$t_0 = 0, y_0 \neq 0, w_0 \quad (1)$$

and non-linear recurrent relations (we denote $\alpha = \gamma c^{-2}$)

$$t_{n+1} - t_n = \Delta_n t = \frac{2y_n + \alpha}{c} \left(1 - \frac{w_n}{c}\right)^{-1} \quad (2)$$

$$y_{n+1} - y_n = \Delta_n y = \alpha + w_n \Delta_n t = \alpha + w_n \frac{2y_n + \alpha}{c} \left(1 - \frac{w_n}{c}\right)^{-1} \quad (3)$$

$$w_{n+1} - w_n = \Delta_n w = \frac{2\alpha}{\Delta_n t} = \frac{c\alpha \left(1 - \frac{w_n}{c}\right)}{y_n + \frac{\alpha}{2}} \quad (4)$$

Interpretations We give a simple interpretation of the formal model as a particle movement along the real line. But firstly we define the simplified model.

At time moments $t \in [0, \infty)$ on the real line there is the fixed particle 1 with coordinate $z_1(t) \equiv 0$ and the particle 2 with coordinate $z_2(t) = y(t) > 0$. There is also the virtual particle 0, interaction transporter, with coordinate $z_0(t)$. It is assumed that for all t

$$z_1(t) \equiv 0 \leq z_0(t) \leq z_2(t) = y(t)$$

Let us assume also that initially at time $t_0 = 0$

$$z_2(0) = z_0(0) = y_0 > 0$$

The particle 0 runs back and forward in-between the particles 1 and 2 with constant sufficiently large velocity c , reflecting from any of them, moreover the coordinate of the particle 2 changes unevenly. During all other time the particle 2 stands still. Denote

$$0 = t_0 < t_1 < \dots < t_n < \dots, \quad (5)$$

the subsequent moments, when the particle 0 collides with particle 2, that is when $z_0(t) = z_2(t) = y(t)$. At these moments the coordinate of particle 2 gets the increment $\alpha > 0$, that is

$$y(t_n) = y(t_n - 0) + \alpha,$$

Thus, we get the following system of recurrent equations for the variables $t_n, y_n, n \geq 0$,

$$\Delta_n y = y_{n+1} - y_n = \alpha, \quad \Delta_n t = t_{n+1} - t_n = \frac{2y_n + \alpha}{c} \quad (6)$$

that has evident solution

$$y_n = y_0 + n\alpha, \quad t_n = \frac{2y_0 + \alpha(1 + 2n)}{c}$$

But more important is that if the particle 2 on the time interval $[t_n, t_{n+1})$ moved under influence of the potential field $E = \frac{\gamma}{y}$, then, starting from the point $y = y_n$ with zero velocity, it would pass the distance α for the time $\Delta t = \Delta_n t$, defined from the equation

$$(F(y) + O(\Delta y)) \frac{(\Delta t)^2}{2} = \alpha$$

From this and from (6) we get:

1) explicit formula for the force and the energy, having newtonian or coulombian form,

$$F(y) \sim \frac{2\alpha}{(\Delta t)^2} = \frac{2\alpha c^2}{(2y_n + \alpha)^2} \sim \frac{\gamma}{2y_n^2} \implies E \sim \frac{\gamma}{2y_n}$$

2) time-energy relationship

$$E \Delta t \sim \frac{\gamma}{c} \quad (7)$$

which could be possibly related to the time-energy quantum uncertainty relation.

This explanation of the inverse square law works only on very small time intervals, and to prove convergence of this model to newtonian trajectory on large time intervals, one needs to introduce velocity of the particle 2 as the effective velocity inherited (the traversed distance to the time necessary for this) from the movement on the previous time interval. Thus, the velocity gets at any time t_n positive increment defined by formula (4). At other time moments one can assume the velocity $w(t)$ of particle 2 constant and equal to w_n on all interval $[t_n, t_{n+1})$. The coordinate of particle 2 increases linearly on this interval. At the moments t_n the functions $y(t)$ and $w(t)$ will be assumed right continuous.

Then it is evident that

$$\Delta_n t = \frac{2y_n + \alpha + w_n \Delta_n t}{c}$$

from where the formula (2) follows. The factor 2 in the formula (4) for the velocity jump could produce some ambiguity. In fact, we can consider that the increment α of the coordinate on the previous step appeared not by itself, but as a result of linear increase of the velocity increment (from 0 to $\frac{2\alpha}{\Delta t}$) on the previous step. But the following interpretation is even more useful.

This interpretation differs from the previous one only by the dynamics inside the intervals (t_n, t_{n+1}) . Moreover, $y(t) = z_2(t)$ and $w(t)$ become continuous, that is there is no jump α at times t_n and no jump of the velocity. However, we define the acceleration $a(t)$ with jumps at times t_n , piece-wise constant, left continuous, constant and equal to a_n on the interval $[t_n, t_{n+1})$.

The parameter α still exists, but will have another meaning. Namely, we assume that particle 2 on the time interval (t_n, t_{n+1}) , having constant acceleration a_n (or constant force $f_n = a_n$), has to pass additional distance α for the time $\Delta_n t = t_{n+1} - t_n$, then

$$a_n \frac{(\Delta_n t)^2}{2} = \alpha \implies a_n = \frac{2\alpha}{(\Delta_n t)^2}$$

At the same time the velocity will gradually change from w_n to

$$w_{n+1} = w_n + \Delta_n w, \Delta_n w = a_n \Delta_n t$$

Thus, we modify the recurrent formulas (2-4) as follows

$$\Delta_n t = \frac{2y_n + w_n \Delta_n t + a_n \frac{(\Delta_n t)^2}{2}}{c}, \quad (8)$$

$$\Delta_n y = w_n \Delta_n t + a_n \frac{(\Delta_n t)^2}{2}, \quad (9)$$

$$\Delta_n w = a_n \Delta_n t, \quad (10)$$

$$a_n = \frac{2\alpha}{(\Delta_n t)^2} \quad (11)$$

that after excluding the acceleration gives the same formulas (2-4). But now, from these formulas we get in addition the expression for the force on the interval (y_n, y_{n+1})

$$f_n = a_n = \frac{2\alpha}{(\Delta_n t)^2} = \frac{2\gamma}{(2y_n + \alpha)^2} \left(1 - \frac{w_n}{c}\right)^2 = \frac{\gamma}{2y_n^2} + D_1 c^{-1} \quad (12)$$

$$D_1 = \frac{2\gamma}{(2y_n + \alpha)^2} \left(-2w_n + \frac{w_n^2}{c}\right) - c\gamma\alpha \frac{4y_n + \alpha}{2y_n^2(2y_n + \alpha)^2}$$

which has, up to $O(\frac{1}{c})$, has inverse square form and indicates on the relation with newtonian dynamics.

Possible modification of the model could be to introduce an additional restriction - kinetic energy conservation in collisions of particles 2 and 0. But it seems that such restriction does not give much new.

Note that similar approach in two particle or even N particle case does not necessitate detailed space structure, but only the distances between each pair of particles. This brings together physics and systems of interacting automata, see ([7, 5, 6, 4]).

In a possible future theory (with discrete space-time or even without any space-time) $\alpha, c > 0$ could be fundamental constants such that $\alpha > 0$ is sufficiently small, and c is sufficiently large.

We will prove that our model converges to the newtonian trajectories of the particle 2 in the scaling limit

$$\alpha = \gamma c^{-2}, c \rightarrow \infty \quad (13)$$

Asymptotic behavior The following result shows that the velocity of particle 2 does not exceed c , thus the virtual particle always catches up particle 2,

Lemma 1 *If $|w_0| < c$, then there exists a unique solution of the system (2-4) with finite t_n, w_n, y_n for all $n > 0$. Moreover, $|w_n| < c$ for all n .*

The following result describes qualitatively the asymptotic behavior of the model $M_0(c, \gamma)$ as $n \rightarrow \infty$.

Lemma 2 *For any fixed $\gamma, y_0 > 0, w_0 \geq 0$, and sufficiently large $c > 0$ (that is for $c > c_0$ for some $c_0 = c_0(\gamma, w_0, y_0)$) we have:*

1) $w_n < c$ for all n ;

2) the sequences $t_n, y_n, \Delta_n t, \Delta_n y$ are strictly increasing to infinity as $n \rightarrow \infty$. The sequence $\Delta_n w$ strictly decreases to zero, and sequence w_n increases to some finite limit $w_\infty \leq c$.

If $\gamma, y_0 > 0, w_0 < 0$, then

3) there exists $N < \infty$ such that $w_N > 0, y_N > 0$, and for $n < N$ we have: $y_n > 0$ and decrease, t_n increase, $w_n < 0$ and increase.

However, convergence to classical mechanics occurs on the scales, where n is of the order c .

Lemma 3 *For any $y_0 > 0, w_0 \geq 0, n \leq Ac$, there exist constants $B_i = B_i(y_0, w_0, A) > 0, i = 1, 2$, such that uniformly in $n \leq Ac$, we have the following upper bounds*

$$t_n, y_n, w_n \leq B_1 < \infty \quad (14)$$

$$\Delta_n t, \Delta_n y, \Delta_n w \leq B_2 c^{-1} \quad (15)$$

and the following lower bounds for any $n > Ac$

$$\Delta_n t \geq \frac{2y_0}{c} \implies t_n \geq 2y_0 A \quad (16)$$

$$y_0 \leq y_n, \alpha + w_0 \frac{2y_0}{c} \leq \Delta_n y \quad (17)$$

$$w_0 \leq w_n, \frac{2\alpha}{\Delta_n t} \leq \frac{2\alpha c}{B_2} = \frac{2\gamma}{B_2 c} \leq \Delta_n w \quad (18)$$

Assume now $w_0 < 0$. Then the integer N , defined in lemma 2, has the bound $N \leq B_4 c$, where

$$B_4 = \frac{|w_0|}{\gamma} \left(y_0 + \frac{\alpha}{2} \right)$$

For $n > N$ the dynamics is similar to the case of positive initial velocity with initial data at $n = N$.

Main result Consider also the trajectory $x(s) \in R$ of the newtonian particle (interacting with another particle fixed at zero), defined by the equation

$$\frac{d^2x}{dt^2} = \frac{\gamma}{2x^2}$$

and initial data

$$x(0) = y_0 > 0, v(0) = \frac{dx}{ds}(0) = w_0$$

We assume $\gamma > 0$ (repulsion). Then as $s \rightarrow \infty$ (monotonically for $w_0 \geq 0$)

$$x(s) \rightarrow \infty, v(s) \rightarrow \sqrt{v^2(0) + \frac{\gamma}{x(0)}}$$

Theorem 1 For given c let $y(t) = y(t, c)$ be arbitrary curve such that for any n and any $t_n \leq t \leq t_{n+1}$

$$y_n \leq y(t) \leq y_{n+1}$$

Then for any interval $I = [0, A]$ there exists $B = B(A)$ such that for sufficiently large $c > c_0(A)$

$$|y(t) - x(t)| \leq Bc^{-1}$$

uniformly in $t \leq t_{[Ac]}$. Then of course for any t and as $c \rightarrow \infty$

$$y(t) \rightarrow x(t)$$

uniformly on any finite interval.

Moreover for any function $w(t)$ such that $w_n \leq w(t) \leq w_{n+1}$ for any n, t ,

$$|w(t) - v(t)| \leq Bc^{-1}, w(t) \rightarrow_{c \rightarrow \infty} v(t)$$

3 Proofs

Proof of lemmas 1 and 2 1) Note first that from formulas (2)-(4) it follows that for $w_0 \geq 0$ all increments $\Delta_n t, \Delta_n y, \Delta_n w$ are positive while $w_n < c$. Moreover, also the sequences t_n, y_n are increasing.

Assume then $y_n, w_n > 0$ for given n and let us show that the following inequality cannot occur

$$w_n < c \leq w_{n+1}$$

Putting $w_{n+1} = c + \epsilon, \epsilon \geq 0$, from (4) we have

$$w_{n+1} = c + \epsilon = w_n + \frac{c\alpha - \alpha w_n}{y_n + \frac{\alpha}{2}}$$

from where we have

$$w_n = \frac{(c + \epsilon)(y_n + \frac{\alpha}{2}) - c\alpha}{(y_n + \frac{\alpha}{2} - \alpha)} = \frac{c(y_n - \frac{\alpha}{2}) + \epsilon(y_n + \frac{\alpha}{2})}{(y_n - \frac{\alpha}{2})} = c + \frac{\epsilon(y_n + \frac{\alpha}{2})}{y_n - \frac{\alpha}{2}}$$

This gives contradiction: if $\epsilon > 0$, then $w_n > c$, and if $\epsilon = 0$, then $w_n = c$. By induction we get that this holds for all n .

2) Also the sequence w_n is increasing, then $\Delta_n t$ also increase by formula (2), and $\Delta_n y$ increase by (3). Then t_n and y_n tend to infinity as $n \rightarrow \infty$. The sequence $\Delta_n w$ is decreasing by formula (4) and because $\Delta_n t$ increase, it tends to zero via last equality in (4). That is why $\Delta_n t \rightarrow \infty$. It follows that w_n tends to a finite limit by property 1).

3) If $w_0 < 0$, let N be such that $y_n > 0, w_n < 0$ for any $n < N$, and one of the following conditions holds:

- a) $y_N > 0, w_N > 0$;
- b) $y_N \leq 0$;
- c) $N = \infty$, that is $y_n > 0, w_n < 0$ for all n .

Let us prove that case b) is impossible. Note first that $\Delta_n t > 0$, and so $\Delta_n w > 0$. while

$$y_n > 0, w_n < 0 \quad (19)$$

that follows from formulas (2) and (4) correspondingly.. Then w_n , under these conditions, increases (decreases in absolute value). Then let

$$y_{n+1} = y_n + \alpha + w_n \frac{2y_n + \alpha}{c} (1 - \frac{w_n}{c})^{-1} < 0$$

or

$$y_n (1 + \frac{2w_n}{c} (1 - \frac{w_n}{c})^{-1}) < -\alpha - \frac{w_n \alpha}{c} (1 - \frac{w_n}{c})^{-1} = -\alpha (1 + \frac{w_n}{c} (1 - \frac{w_n}{c})^{-1})$$

which is impossible as the right hand side part is negative (as from $|w_0| < c$ it follows that $|w_n| < c$), and the left hand side (as $y_n > 0$).

Let us prove now that case c) is impossible. There can be only three cases:

c1) y_n decrease, or more generally, tend to some limit, or even more general, for some sequence n_k the numbers $\Delta_{n_k} t$ are uniformly bounded from above by a positive constant. The $\Delta_{n_k} w$, by (4), are bounded from below by some positive constant. Then the velocity becomes non-negative on some step. Such case has already been considered.

c2) As we have shown, w_n tend to some limit w . Firstly, let $w < 0$. Then $\Delta_n w \rightarrow 0$, that implies $\Delta_n t \rightarrow \infty$ and, by (3), $\Delta_n y$ asymptotically behaves like $w \Delta_n t$, and then y_n will become negative, that is impossible as we have shown earlier.

c3) w_n tend to the limit $w = 0$. Then, as $0 < \Delta_n w < |w_n|$,

$$\Delta_n y = \alpha + w_n \Delta_n t = \alpha + 2\alpha \frac{w_n}{\Delta_n w} \leq -\alpha \quad (20)$$

from where we get that y_n will become negative, that is impossible.

Let us finish now the proof of point 3). In case a) it was shown above what occurs after the step N when w_N became positive. Let us see now what was before this. We have $\Delta_n t > 0$ by (2), it follows that t_n increase, and $\Delta_n w > 0$ by (4). At the same time $y_n > 0$ by definition and decrease. As by (20) $\Delta_n y < -\alpha$, thus y_n decrease.

Proof of Lemma 3 Firstly consider the case $w_0 \geq 0$. We shall get first an upper bound for w_n linear in A . As $\Delta_n w$ decrease, then from the last equality of the formula (4) it follows that for $n \leq Ac$

$$w_n = w_0 + \sum_{k=0}^{n-1} \Delta_k w \leq w_0 + Ac \max_{k=0}^{n-1} \Delta_k w \leq w_0 + Ac \Delta_0 w \leq B_3 = w_0 + \frac{A\alpha c^2}{y_0} = w_0 + \frac{A\gamma}{y_0} \quad (21)$$

where the last inequality follows from the last equality of the formula (4).

Formula (2) gives evident linear lower bound for t_n . And moreover

$$\begin{aligned}
\Delta_n y &= \alpha + w_n \Delta_n t \leq \alpha + B_3 \Delta_n t \leq \alpha + \frac{2B_3}{c} y_n = \\
&= \frac{2B_3}{c} \left(\frac{\alpha c}{2B_3} + y_0 + \Delta_0 y + \dots + \Delta_{n-1} y \right) \leq \frac{2B_3}{c} \left(1 + \frac{2B_3}{c} \right) (y'_0 + \Delta_0 y + \dots + \Delta_{n-2} y) \leq \\
&\leq \frac{2B_3}{c} y'_0 \left(1 + \frac{2B_3}{c} \right)^n \leq \frac{2B_3 y'_0}{c} \left(1 + \frac{2B_3}{c} \right)^{Ac} \leq \frac{2B_3 y'_0}{c} e^{\frac{A}{2B}}
\end{aligned} \tag{22}$$

where we denoted $y'_0 = \frac{\alpha c}{2B_3} + y_0$. Then

$$y_n = y_0 + \Delta_0 y + \dots + \Delta_{n-1} y \leq 2B_3 y_0 e^{\frac{A}{2B}} \frac{n_3}{c} \tag{23}$$

$$\Delta_n t \leq \frac{2y_n}{c} \implies t_n \leq 2y_n A \tag{24}$$

Consider now the case of negative initial velocity $w_0 < 0$. Let N be the first n , when w_n becomes positive. Then from the lower bound for $\Delta_n w$

$$\Delta_n w = \frac{c\alpha(1 - \frac{w_n}{c})}{y_n + \frac{\alpha}{2}} \geq \frac{c\alpha}{y_n + \frac{\alpha}{2}} \geq \frac{c\alpha}{y_0 + \frac{\alpha}{2}} = \frac{1}{c} \frac{\gamma}{y_0 + \frac{\alpha}{2}}$$

we get $N \leq B_4 c$. where

$$B_4 = \frac{|w_0|}{\gamma} \left(y_0 + \frac{\alpha}{2} \right)$$

Note also that from (20) we know that $\Delta_n y \leq -\alpha$, and moreover it is easy to see that

$$\begin{aligned}
\Delta_n t &\leq \frac{2y_0 + \alpha}{c}, t_N \leq B_4(2y_0 + \alpha) \\
0 &\leq w_N \leq \Delta_{N-1} w
\end{aligned}$$

Bundle of Hamiltonian systems Further on it we will introduce some functions denoted by $D_i = D_i(y_0, w_0, c, n)$, $i = 1, 2, \dots, 10$, which depend on y_0, w_0, n, c . We will see that there exist constant $c_0 = c_0(y_0, w_0) > 0$ such that these functions are uniformly bounded for given y_0, w_0 and all c, n such that $n < Ac$, $c > c_0$.

The introduced dynamics is not hamiltonian, as the force f_n depends on the initial conditions y_0, w_0 (through y_n, w_n). However, we will define the bundle $H = H_{y_0, w_0}$ of hamiltonian systems, which depend on y_0, w_0 , and also of c, γ , as of parameters. Then we shall prove that any H_{y_0, w_0} approximates the newtonian dynamics with the same initial data. However for other initial data H_{y_0, w_0} can exhibit quite different behavior.

First of all, define the force $f(y)$ (depending of the same initial data) on all time interval, putting it equal to f_n on the interval $[y_n, y_{n+1})$.

This force defines, for any $y \in [y_n, y_{n+1})$, the potential $V(y)$, velocity $w(y)$ and kinetic energy $W(y)$, as follows

$$V(y) = - \int_{y_0}^y f(x) dx = - \sum_{k=0}^n f_k \Delta_k y - f_n (y - y_n) \tag{25}$$

$$w(y) = w_n + f_n(t - t_n), W(y) = \frac{w^2(y)}{2}$$

Lemma 4

$$V(y) = -\frac{\gamma}{2y_0} + \frac{\gamma}{2y} + D_2 c^{-1} \quad (26)$$

Proof. Let us rewrite (25), using (11) as (2), as

$$\begin{aligned} V(y) &= -\sum_{k=0}^n a_k \Delta_k y + D_3 c^{-1} = -\sum_{k=0}^n \frac{2\alpha}{(\Delta_k t)^2} \Delta_k y + D_3 c^{-1} = -\sum_{k=0}^n \left(\frac{\gamma}{2y_n^2} + D_1 c^{-1} \right) \Delta_k y + D_3 c^{-1} = \\ &= -\frac{\gamma}{2} \sum_{k=0}^n \left(\frac{1}{y_k^2} \right) \Delta_k y + D_5 c^{-1} = -\frac{\gamma}{2} \sum_{k=0}^n \left(\frac{1}{y_k} - \frac{1}{y_{k+1}} + D_6 c^{-2} \right) + D_5 c^{-1} = \\ &= -\frac{\gamma}{2y_0} + \frac{\gamma}{2y} + D_2 c^{-1} \end{aligned} \quad (27)$$

as

$$\frac{1}{y_k} - \frac{1}{y_{k+1}} = \frac{\Delta_k y}{y_k y_{k+1}} = \frac{\Delta_k y}{y_k (y_k + \Delta_k y)} = \frac{\Delta_k y}{y_k^2} + D_6 c^{-2} \quad (28)$$

Here

$$|D_3 c^{-1}| \leq a_n \Delta_n t, \quad |D_5 c^{-1}| \leq |D_1 c^{-1}| y_n + |D_3 c^{-1}|$$

$$|D_6 c^{-2}| \leq, \quad |D_2 c^{-1}| \leq \frac{\gamma}{2} |D_6| c^{-2} y_n + |D_5| c^{-1}$$

Proof of the theorem Let s, v, x be the time, velocity and the coordinate of the newtonian particle satisfying the equation

$$\frac{d^2 x}{ds^2} = \frac{\gamma}{2x^2} \quad (29)$$

with initial data $s_0 = 0, v(0), x(0)$. Denote kinetic and potential energy of the newtonian particle as

$$T = \frac{v^2}{2}, U = \frac{\gamma}{2x} - \frac{\gamma}{2y_0} \quad (30)$$

correspondingly (here $-\frac{\gamma}{2y_0}$ is just a convenient constant).

We will compare dynamics of two hamiltonian systems - newtonian one and H_{y_0, w_0} with the same initial conditions

$$y(0) = x(0) = y_0, w(0) = v(0) = w_0$$

where t, w, y are the time, velocity and coordinate of the particle in the system H_{y_0, w_0} . We will compare times s_n and t_n , also velocities v_n and w_n at the points y_n . Then by (26) and (30)

$$W(y_n) - W(y_0) = V(y_0) - V(y_n) = U(y_0) - U(y_n) + D_{11} c^{-1} = -T(y_0) + T(y_n) + D_1 c^{-1} \quad (31)$$

and hence

$$w(y_n) = v(y_n) + D_8 c^{-1} \quad (32)$$

From the energy conservation law we have at the point $x = x(s)$ (at some, for a while unknown, time s)

$$v^2(s) + \left(\frac{\gamma}{2x} - \frac{\gamma}{2y_0} \right) = v^2(0) \implies \frac{dx}{dt} = v(x(s)) = \sqrt{v^2(0) + \frac{\gamma}{x(0)} - \frac{\gamma}{x}}$$

and similarly at the point $y_n = y(t_n)$ at time t_n

$$w(t_n) = \sqrt{w_0^2 + \frac{\gamma}{y_0} - \frac{\gamma}{y_n} + D_9 c^{-1}}$$

or

$$s = \int_{x_0}^{x(s)} \frac{dx}{\sqrt{v^2(0) + \frac{\gamma}{x(0)} - \frac{\gamma}{x}}}, t = \int_{y_0}^{y(t)} \frac{dy}{\sqrt{w_0^2 + \frac{\gamma}{y_0} - \frac{\gamma}{y} + D_9 c^{-1}}}$$

But as $v(0) = w_0$, $x(0) = y_0$, and we consider the equation $x(s) = y(t)$, we get

$$s = t + D_{10} c^{-1} \tag{33}$$

and hence

$$y(t) = x(s) = x(t) + D_{11} c^{-1} \tag{34}$$

The bounds for D_i , which are indicated in their definition, are simple exercises in analysis, and we omit them.

References

- [1] Gerard 't Hooft. Can there be Physics without Experiments? Challenges and Pitfalls. Revised version of Oscar Klein lecture. Stockholm, June 1, 1999), SPIN-2000/24, Int. Journal of Modern Physics A 16 (2001) 2895-2908.
- [2] Arnold V. I. Huygens and Barrow, Newton and Hooke. 1990. Birkhauser.
- [3] Hall R. W., Josic K. Planetary Motion and the Duality of Force Laws. SIAM Review, 2000, v. 42, No. 1, pp. 115-124.
- [4] Schiff J.L. Cellular Automata. A Discrete View of the World. 2008. Wiley-Interscience.
- [5] Ilachinski A. Cellular Automata. A Discrete Universe. 2001. World Scientific.
- [6] Kantor F. W. Information Mechanics. 1977. Wiley. New York.
- [7] Alur Rajeev, Dill David. A theory of timed automata. Theoretical Computer Science, 1994, 126, 183-235.
- [8] Beardon A. F. Iteration of Rational Functions. Complex analytic dynamical systems. Springer-Verlag. 1990.